ME 6135: Advanced Aerodynamics

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Lecture-15

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Expansion wave

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Prandtl-Meyer Expansion wave

When a supersonic flow is **turned away from itself**, an **expansion wave** is formed as shown in Fig. 1.
This is **directly opposite** to the situation when the flow is **turned into itself**, with the consequent shock wave as shown in Fig. 2.
Expansion waves are the opposites of shock waves.

Some qualitative aspects of expansion waves:

1. $M_2 > M_1$ ⬆

2. $\frac{p_2}{p_1} < 1, \quad \frac{\rho_2}{\rho_1} < 1, \quad \frac{T_2}{T_1} < 1$ ⬇

The Mach number increases while pressure, density, and temperature decrease through an expansion wave.

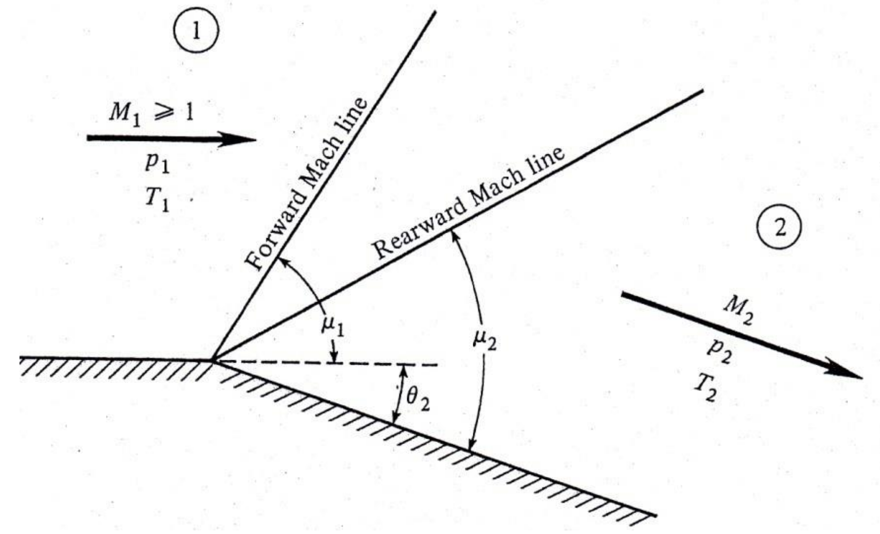


Fig. 1 Prandtl-Meyer Expansion wave

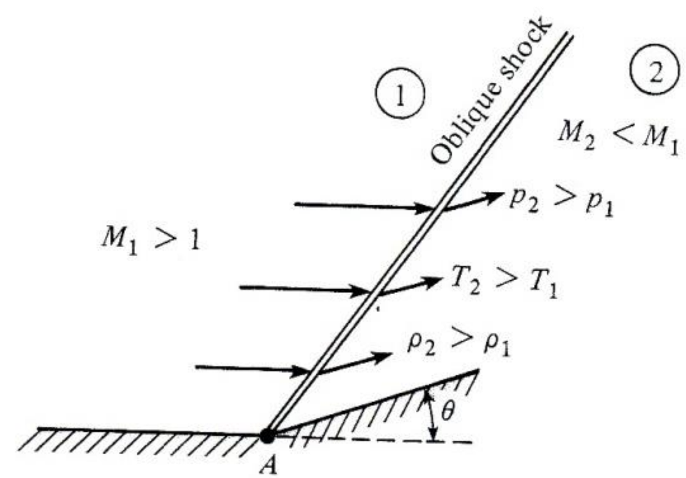


Fig. 2 Oblique shock wave

Prandtl-Meyer Expansion wave

3. The expansion fan is a continuous expansion region; Composed of an infinite number of Mach waves.

Forward Mach line : $$\mu_1 = \sin^{-1}\left(\frac{1}{M_1}\right)$$

Rearward Mach line : $$\mu_2 = \sin^{-1}\left(\frac{1}{M_2}\right)$$

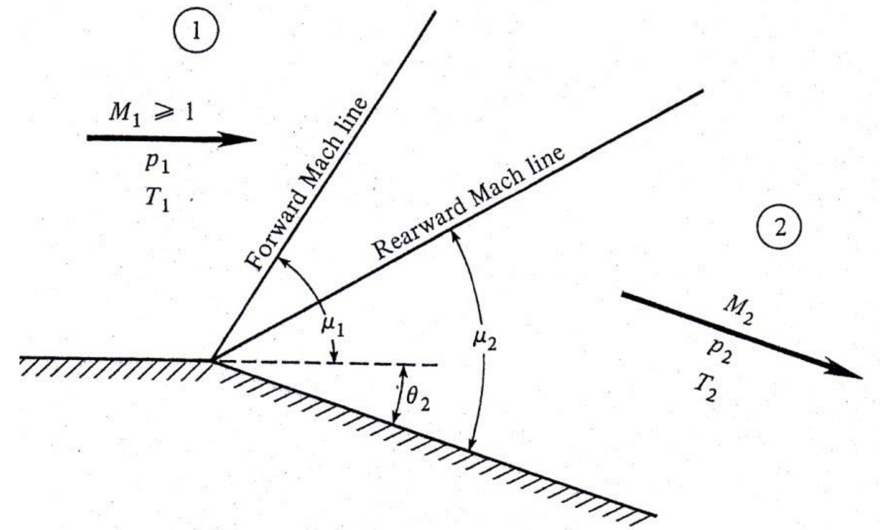
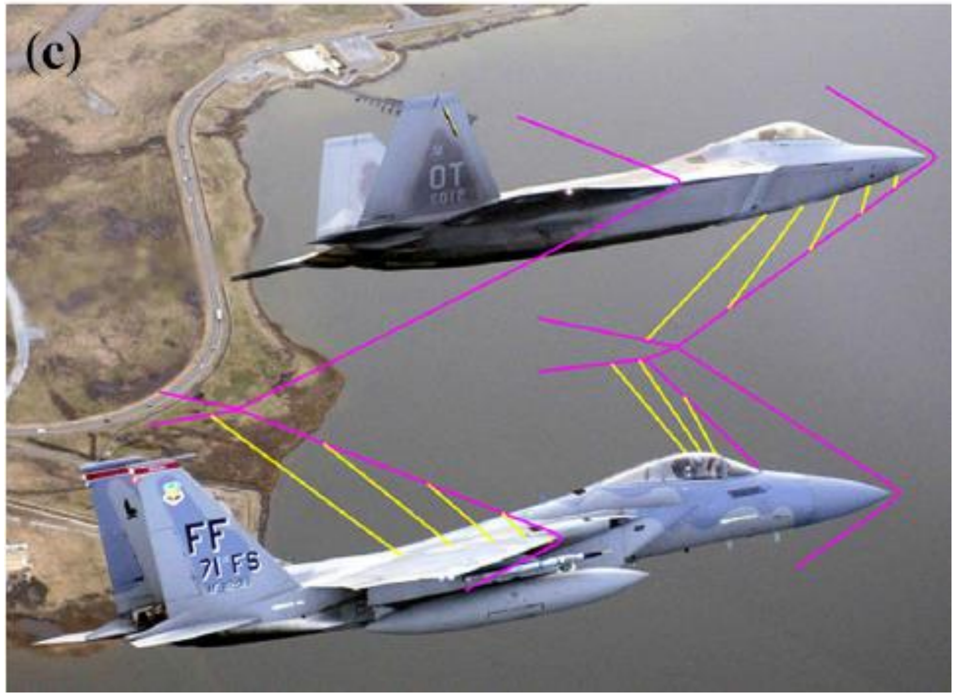


Fig. 1 Prandtl-Meyer Expansion wave

4. Streamlines through an expansion wave are smooth curved lines.

5. since the expansion takes place through a continuous succession of Mach waves, and *ds* = 0 for each Mach wave, the **expansion is isentropic.**

Prandtl-Meyer Expansion wave



Front of the subsonic airplane is rounded while in supersonic aircraft, it is sharp and pointed to avoid detached shock wave development.

Source: Journal of Visualization 2015 (18)

Prandtl-Meyer Expansion wave

The problem of a Prandtl-Meyer expansion wave can be analyzed by considering the infinitesimal changes across a very weak wave (essentially a Mach wave) produced by an infinitesimal small flow deflection $d\vartheta$ as shown in Fig. 4.33.

From the law of sines:

$$\frac{V+dV}{V} = \frac{\sin(\pi/2+\mu)}{\sin(\pi/2-\mu-d\theta)}$$

$$\Rightarrow 1+\frac{dV}{V} = \frac{\cos\mu}{\cos(\mu+d\theta)} \qquad ;\ \text{Trigonometry}$$

$$\Rightarrow 1+\frac{dV}{V} = \frac{\cos\mu}{\cos\mu\cos d\theta - \sin\mu\sin d\theta}$$

$$\Rightarrow 1+\frac{dV}{V} = \frac{\cos\mu}{\cos\mu - d\theta\sin\mu} \qquad ;\ \text{for } d\theta\to 0, \cos d\theta \to 1, \sin d\theta \to d\theta$$

$$\Rightarrow 1+\frac{dV}{V} = \frac{1}{1-d\theta\tan\mu}$$

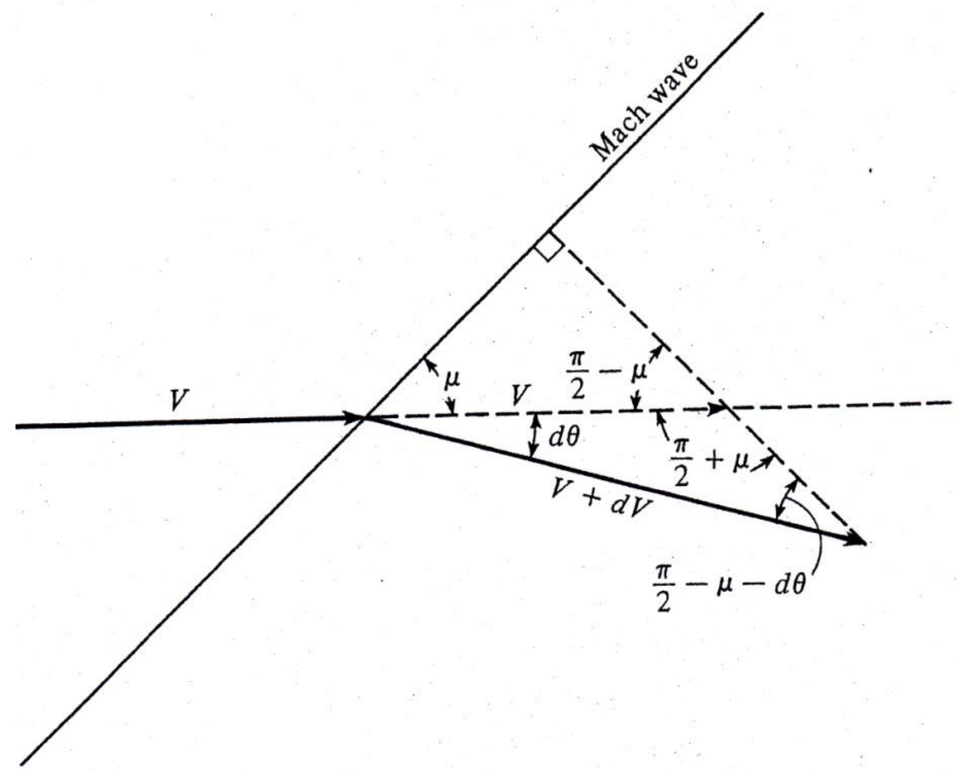


Figure 4.33 | Geometric construction for the infinitesimal changes across a Mach wave; for use in the derivation of the Prandtl–Meyer function. Note that the change in velocity across the wave is normal to the wave.

Prandtl-Meyer Expansion wave

Recalling the series expansion;

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \dots \dots$$

$$\Rightarrow 1 + \frac{dV}{V} = 1 + d\theta \tan\mu + (d\theta \tan\mu)^2 + (d\theta \tan\mu)^3 + \dots$$

$$\Rightarrow 1 + \frac{dV}{V} \approx 1 + d\theta \tan\mu \quad \text{(neglecting HOT)}$$

$$\Rightarrow \frac{dV}{V} = d\theta \tan\mu$$

$$\Rightarrow d\theta = \frac{\frac{dV}{V}}{\tan\mu}$$

$$\Rightarrow d\theta = \sqrt{M^2 - 1}\,\frac{dV}{V}$$

since $\mu = \sin^{-1}\frac{1}{M}$ $\quad \therefore \tan\mu = \frac{1}{\sqrt{M^2-1}}$

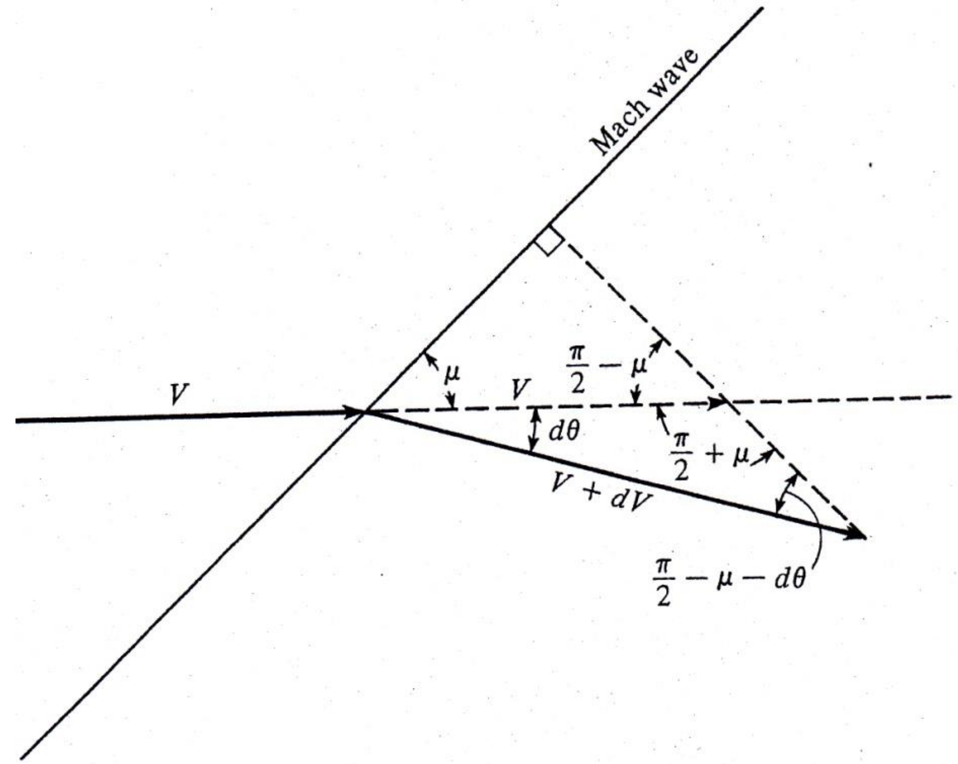


Figure 4.33 | Geometric construction for the infinitesimal changes across a Mach wave; for use in the derivation of the Prandtl–Meyer function. Note that the change in velocity across the wave is normal to the wave.

The governing differential equation for Prandtl-Meyer expansion flow.

Prandtl-Meyer Expansion wave

To analyze the entire Prandtl-Meyer expansion as shown in figure, the governing equation must be integrated over the complete angle, θ_2 as

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\,\frac{dV}{V} \qquad \text{(i)}$$

From the definition of Mach number;

$$V = Ma$$

$$\Rightarrow \ln V = \ln M + \ln a$$

$$\Rightarrow \frac{dV}{V} = \frac{dM}{M} + \frac{da}{a} \qquad \text{(ii)}$$

On differentiation

Again, for a calorically perfect gas, the adiabaic energy equation -

$$\left(\frac{a_0}{a}\right)^2 = \frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

$$\Rightarrow \quad a = a_0 \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1/2} \qquad \text{(iii)}$$

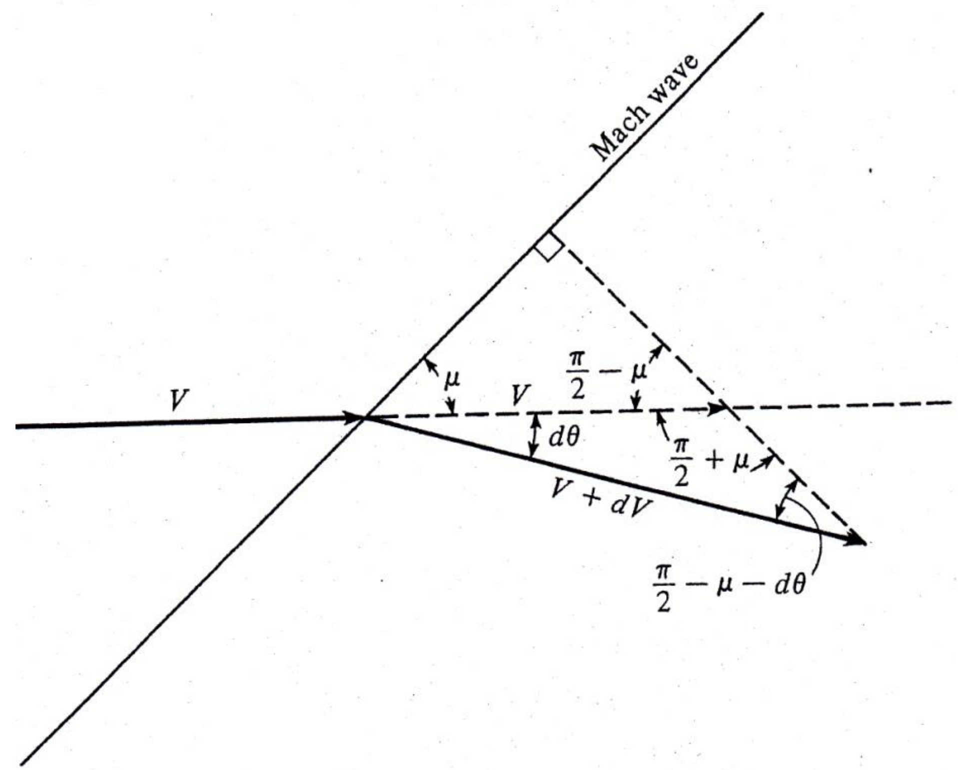


Figure 4.33 | Geometric construction for the infinitesimal changes across a Mach wave; for use in the derivation of the Prandtl–Meyer function. Note that the change in velocity across the wave is normal to the wave.

Prandtl-Meyer Expansion wave

Differentiating the last equation Eq. (iii);

$$\frac{da}{a} = -\left(\frac{\gamma-1}{2}\right)M\left(1+\frac{\gamma-1}{2}M^2\right)^{-1} dM$$

Substituting the above expression in (ii);

$$\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a}$$

$$\Rightarrow \frac{dV}{V} = \frac{1}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

Now, equation (i) can be found as-

$$\int_{\theta_1}^{\theta_2} d\theta = \theta_2 - 0 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M} \ ; \quad \text{horizontal inflow } \theta_1 = 0^o$$

$$\Rightarrow \theta_2 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M} \qquad \text{(iv)}$$

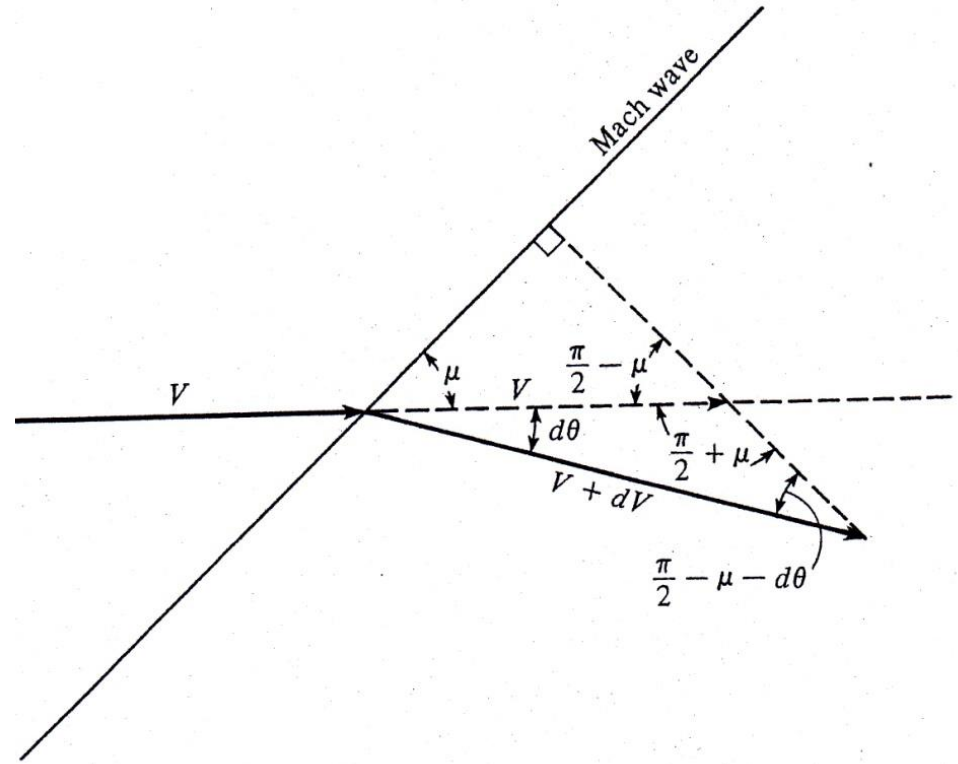


Figure 4.33 | Geometric construction for the infinitesimal changes across a Mach wave; for use in the derivation of the Prandtl–Meyer function. Note that the change in velocity across the wave is normal to the wave.

Prandtl-Meyer Expansion wave

The integral is called the **Prandtl-Meyer function** and is given by symbol ν

$$\nu = \int \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

Performing the integration results:

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}$$

The constant of integration is not important because it drops out while used in complete form of eq. (iii)

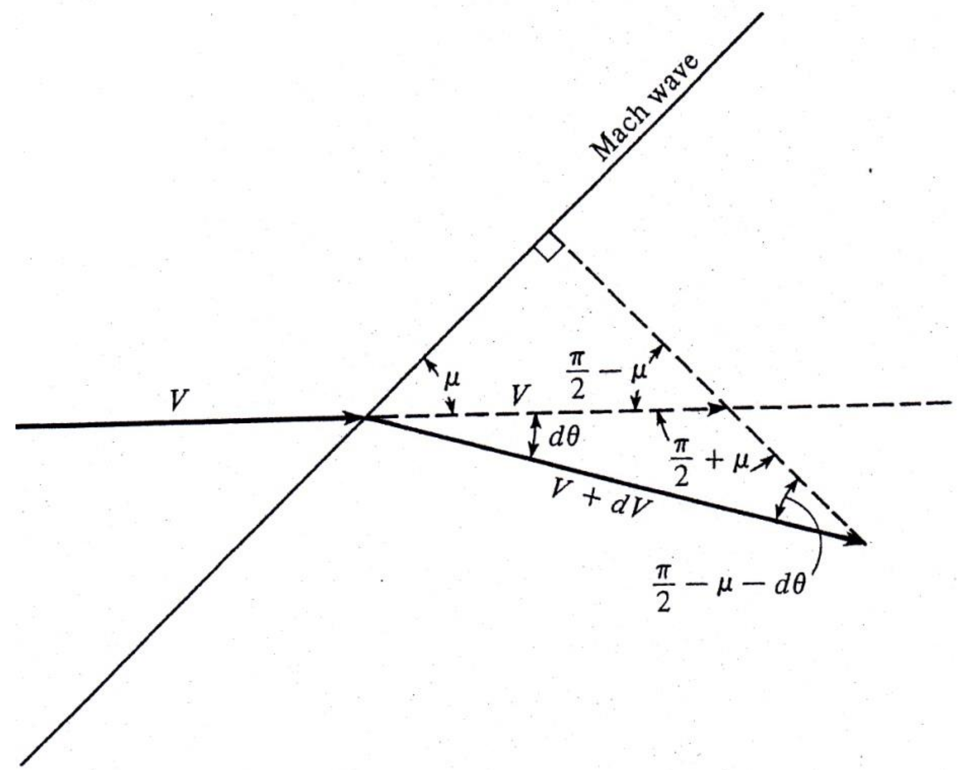


Figure 4.33 | Geometric construction for the infinitesimal changes across a Mach wave; for use in the derivation of the Prandtl–Meyer function. Note that the change in velocity across the wave is normal to the wave.

Finally it could be written as;

$$\theta_2 = \nu(M_2) - \nu(M_1)$$

In case of horizontal inflow, $\theta_1 = 0$

Otherwise

$$\theta_2 - \theta_1 = \nu(M_2) - \nu(M_1)$$

The **Prandtl-Meyer function** is tabulated as a function of Mach number, M in Appendix-C for γ**=1.4**

Prandtl-Meyer Function and Mach Angle

M	ν	μ	M	ν	μ
0.1000 + 01	0.0000	0.9000 + 02	0.1600 + 01	0.1486 + 02	0.3868 + 02
0.1020 + 01	0.1257 + 00	0.7864 + 02	0.1620 + 01	0.1545 + 02	0.3812 + 02
0.1040 + 01	0.3510 + 00	0.7406 + 02	0.1640 + 01	0.1604 + 02	0.3757 + 02
0.1060 + 01	0.6367 + 00	0.7063 + 02	0.1660 + 01	0.1663 + 02	0.3704 + 02
0.1080 + 01	0.9680 + 00	0.6781 + 02	0.1680 + 01	0.1722 + 02	0.3653 + 02
0.1100 + 01	0.1336 + 01	0.6538 + 02	0.1700 + 01	0.1781 + 02	0.3603 + 02
0.1120 + 01	0.1735 + 01	0.6323 + 02	0.1720 + 01	0.1840 + 02	0.3555 + 02
0.1140 + 01	0.2160 + 01	0.6131 + 02	0.1740 + 01	0.1898 + 02	0.3508 + 02
0.1160 + 01	0.2607 + 01	0.5955 + 02	0.1760 + 01	0.1956 + 02	0.3462 + 02
0.1180 + 01	0.3074 + 01	0.5794 + 02	0.1780 + 01	0.2015 + 02	0.3418 + 02
0.1200 + 01	0.3558 + 01	0.5644 + 02	0.1800 + 01	0.2073 + 02	0.3375 + 02
0.1220 + 01	0.4057 + 01	0.5505 + 02	0.1820 + 01	0.2130 + 02	0.3333 + 02
0.1240 + 01	0.4569 + 01	0.5375 + 02	0.1840 + 01	0.2188 + 02	0.3292 + 02
0.1260 + 01	0.5093 + 01	0.5253 + 02	0.1860 + 01	0.2245 + 02	0.3252 + 02
0.1280 + 01	0.5627 + 01	0.5138 + 02	0.1880 + 01	0.2302 + 02	0.3213 + 02
0.1300 + 01	0.6170 + 01	0.5028 + 02	0.1900 + 01	0.2359 + 02	0.3176 + 02
0.1320 + 01	0.6721 + 01	0.4925 + 02	0.1920 + 01	0.2415 + 02	0.3139 + 02
0.1340 + 01	0.7279 + 01	0.4827 + 02	0.1940 + 01	0.2471 + 02	0.3103 + 02
0.1360 + 01	0.7844 + 01	0.4733 + 02	0.1960 + 01	0.2527 + 02	0.3068 + 02
0.1380 + 01	0.8413 + 01	0.4644 + 02	0.1980 + 01	0.2583 + 02	0.3033 + 02

Prandtl-Meyer Expansion wave

The calculation of expansion wave is as follows:

1. $\nu(M_1)$ from Appendix-C for the given M_1
2. $\nu(M_2) = \theta_2 + \nu(M_1)$
3. M_2 from Appendix-C
4. The expansion is **isentropic** $\therefore T_0, P_0$ are constant through the wave

$\therefore$

$$\frac{T_1}{T_2} = \frac{1 + \frac{\gamma - 1}{2} M_2^2}{1 + \frac{\gamma - 1}{2} M_1^2}$$

$$\frac{p_1}{p_2} = \left[\frac{1 + \frac{\gamma - 1}{2} M_2^2}{1 + \frac{\gamma - 1}{2} M_1^2} \right]^{\frac{\gamma}{\gamma - 1}}$$

Prandtl-Meyer Expansion wave

EXAMPLE 9.9

A supersonic flow with $M_1 = 1.5$, $p_1 = 1$ atm, and $T_1 = 288$ K is expanded around a sharp corner (see Figure 9.26) through a deflection angle of 15°. Calculate M_2, p_2, T_2, $p_{0,2}$, $T_{0,2}$, and the angles that the forward and rearward Mach lines make with respect to the upstream flow direction.

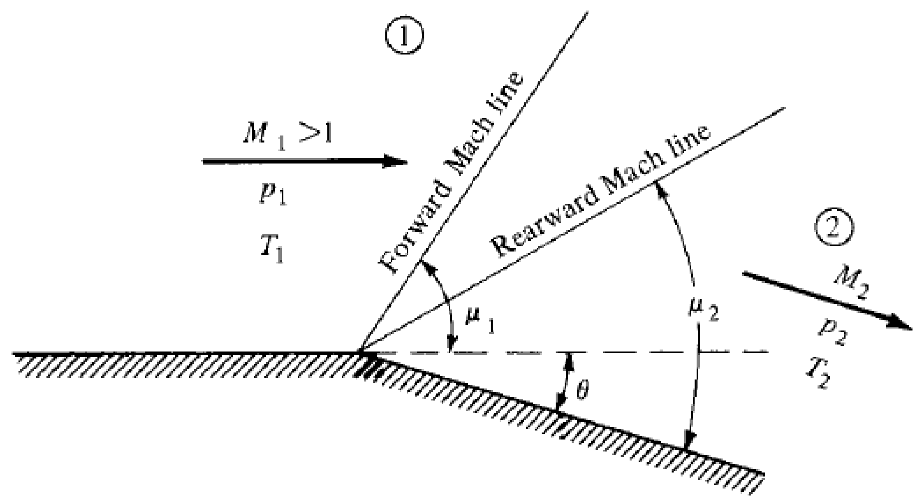


Figure 9.26 Prandtl-Meyer expansion.

■ **Solution**

From Appendix C, for $M_1 = 1.5$, $\nu_1 = 11.91°$. From Equation (9.43), $\nu_2 = \nu_1 + \theta = 11.91 + 15 = 26.91°$. Thus, $\boxed{M_2 = 2.0}$ (rounding to the nearest entry in the table).

From Appendix A, for $M_1 = 1.5$, $p_{0,1}/p_1 = 3.671$ and $T_{0,1}/T_1 = 1.45$, and for $M_2 = 2.0$, $p_{0,2}/p_2 = 7.824$ and $T_{0,2}/T_2 = 1.8$.

Since the flow is isentropic, $T_{0,2} = T_{0,1}$ and $p_{0,2} = p_{0,1}$. Thus,

$$p_2 = \frac{p_2}{p_{0,2}}\frac{p_{0,2}}{p_{0,1}}\frac{p_{0,1}}{p_1}p_1 = \frac{1}{7.824}(1)(3.671)(1\text{ atm}) = \boxed{0.469\text{ atm}}$$

$$T_2 = \frac{T_2}{T_{0,2}}\frac{T_{0,2}}{T_{0,1}}\frac{T_{0,1}}{T_1}T_1 = \frac{1}{1.8}(1)(1.45)(288) = \boxed{232\text{ K}}$$

$$p_{0,2} = p_{0,1} = \frac{p_{0,1}}{p_1}p_1 = 3.671(1\text{ atm}) = \boxed{3.671\text{ atm}}$$

$$T_{0,2} = T_{0,1} = \frac{T_{0,1}}{T_1}T_1 = 1.45(288) = \boxed{417.6\text{ K}}$$

Returning to Figure 9.26, we have

$$\text{Angle of forward Mach line} = \mu_1 = \boxed{41.81°}$$

$$\text{Angle of rearward Mach line} = \mu_2 - \theta = 30 - 15 = \boxed{15°}$$

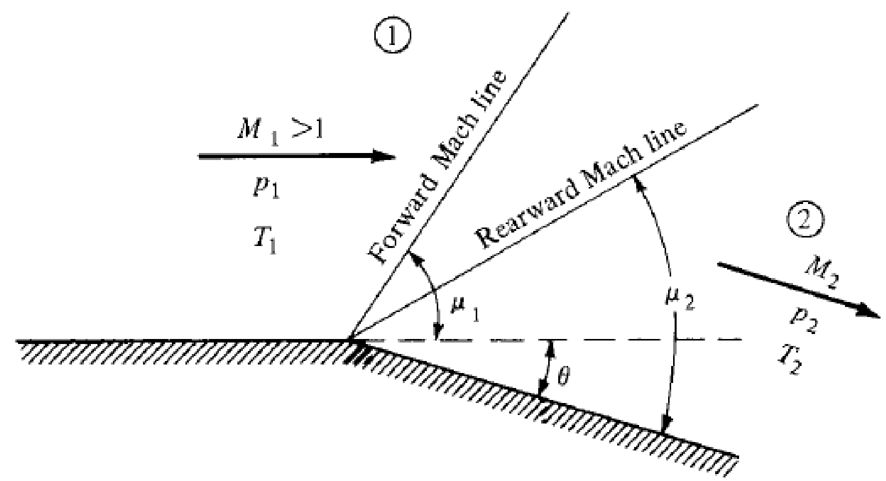


Figure 9.26 Prandtl-Meyer expansion.

Tutorial # 2

11/01/2025 SAT

Time: 7:00 PM

Syllabus: Oblique shock wave, Expansion wave